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Generalized quantum potentials

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Abstract
We first recall that the system of fluid mechanics equations (Euler and continuity) that describes a fluid in irrotational motion subjected to a generalized quantum potential (depending on a constant which may be different from the standard quantum constant $\hbar$) is equivalent to a generalized Schrödinger equation for a 'wavefunction' whose modulus squared yields the fluid density. Then we show that even in the case of the presence of vorticity, it is also possible to obtain a nonlinear Schrödinger-like equation (now of the vectorial field type) from the continuity and Euler equations including a quantum potential. The same kind of transformation also applies to a classical charged fluid subjected to an electromagnetic field and to an additional potential having the form of a quantum potential. Such a fluid can therefore be described by an equation of the Ginzburg–Landau type, and is expected to show some superconducting-like properties. Moreover, a Schrödinger form can be obtained for a fluctuating rotational motion of a solid. In this case the mass is replaced by the tensor of inertia, and a generalized form of the quantum potential is derived. We finally reconsider the case of a standard diffusion process, and we show that, after a change of variable, the diffusion equation can also be given the form of a continuity and Euler system including an additional potential energy. Since this potential is exactly the opposite of a quantum potential, the quantum behavior may be considered, in this context, as an anti-diffusion.

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1. Introduction

It has been known since the origins of quantum mechanics [1] that the Schrödinger equation,

$$\mathcal{D}^2 \Delta \psi + \mathrm{i}\mathcal{D} \frac{\partial \psi}{\partial t} - \frac{\phi}{2m} \psi = 0, \tag{1}$$

where $\mathcal{D} = \hbar/2m$, and $\psi = \sqrt{P}\,\mathrm{e}^{\mathrm{i}\theta}$, can be given the form of a fluid-like system of equations in terms of a probability density $P = |\psi|^2$ and of a velocity $V = 2\mathcal{D}\nabla\theta$. The imaginary and real parts of the Schrödinger equation respectively yield a continuity equation and an Euler-like equation (after differentiation)

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0, \qquad \left(\frac{\partial}{\partial t} + V \cdot \nabla\right) V = -\nabla\left(\frac{\phi + Q}{m}\right), \tag{2}$$

including an additional 'quantum potential' Q, that reads[1]

$$Q = -2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{3}$$

The aim of the present paper is to study the reverse transformation and to suggest generalizations of the possible forms of quantum potentials. The question asked is whether a classical fluid of density ρ subjected to a quantum-like potential $Q = -2\mathcal{D}^2\Delta\sqrt{\rho}/\sqrt{\rho}$ can be described by a Schrödinger-type equation for a 'wavefunction' ψ such that $\rho \propto |\psi|^2$.

As we shall see, the answer is positive. Generalized forms of the quantum potential can still be found, and the application of such quantum potentials to classical Euler and continuity equations allows one to integrate them into Schrödinger equations, possibly of the nonlinear type. Moreover, this transformation can be done for any value of the coefficient $\mathcal{D}$, which is no longer restricted to the standard quantum value $\mathcal{D} = \hbar/2m$ and can take macroscopic values, therefore leading to macroscopic Schrödinger-type equations.

For the construction of these generalizations (in particular to a tensorial form in section 4) and for the interpretation and possible application of these results to observational data and experimental devices, we shall be helped by a third representation of the same equations, the geodesic form obtained in the scale relativity theory [2, 5, 6, 28]. In this framework one constructs, as a manifestation of fractal geometry and of local irreversibility, a covariant derivative $\widehat{\mathrm{d}}/\mathrm{d}t = \partial/\partial t + \mathcal{V}\cdot\nabla - \mathrm{i}\mathcal{D}\Delta$ which allows one to write the equation of motion under the free Galilean form, $\widehat{\mathrm{d}}\mathcal{V}/\mathrm{d}t = 0$. This equation can then be integrated under the form of a Schrödinger equation [2, chapter 5.6].

A fluid that would be subjected to this kind of generalized quantum-like potential is expected to exhibit some macroscopic properties typical of quantum fluids (though certainly not every aspects of a genuine quantum system). We then conclude by a discussion of the possible ways by which such a new type of classical fluid owning some quantum-type properties could be either identified in natural systems, or achieved in experimental and technological devices.

2. The scale relativity approach to a Schrödinger equation

2.1. Dynamics equation in a fractal space

We give in this section a summary of the way a Schrödinger equation is derived in the scale relativity framework, in which spacetime is described as a non-differentiable (therefore fractal) continuum (see e.g. [7] for more detail). Although not absolutely necessary in the context of the present work (which studies the effect of adding quantum potentials to the equations of classical fluids), this approach is actually revealed to be useful for (i) constructing tensorial generalizations of quantum potentials (section 4) and (ii) suggesting real situations and

[1] The quantum potential Q and the velocity field V are well defined only for nonzero ψ. When $P \to 0$, Q diverges when $\Delta\sqrt{P} \neq 0$, so that the two representations are not fully equivalent. It is an open problem how to treat nodal surfaces separating domains: this problem lies outside the scope of the present paper (but see section 6) and will be considered in a forthcoming work.

systems (natural or artificial) in which the effects studied here could be effectively achieved (section 6).

A nondifferentiable geometry implies three main consequences [2, 4, 5, 7]:

(i) The number of fractal geodesics is infinite. This leads to the adoption of a generalized statistical fluid-like description where the velocity $V(t)$ is replaced by a scale-dependent velocity field $V[X(t,\mathrm{d}t),t,\mathrm{d}t]$ (3D case, see [5] for the 4D generalization).

(ii) There is a breaking of the reflexion invariance on the differential element $\mathrm{d}t$. Indeed, in terms of fractal functions $f(t,\mathrm{d}t)$ which are explicitly dependent on the differential element $\mathrm{d}t$ [2], two derivatives are defined,

$$X'_{+}(t,\mathrm{d}t) = \frac{X(t+\mathrm{d}t,\mathrm{d}t) - X(t,\mathrm{d}t)}{\mathrm{d}t}, \qquad X'_{-}(t,\mathrm{d}t) = \frac{X(t,\mathrm{d}t) - X(t-\mathrm{d}t,\mathrm{d}t)}{\mathrm{d}t}, \quad (4)$$

which transform one into the other under the reflection ($\mathrm{d}t \leftrightarrow -\mathrm{d}t$), and which have *a priori* no reason to be equal. This leads to a fundamental two-valuedness of the velocity field.

(iii) The geodesics are themselves fractal curves of fractal dimension $D_F = 2$ [8].

This means that one defines two divergent fractal velocity fields, $V_+[x(t,\mathrm{d}t),t,\mathrm{d}t] = v_+[x(t),t] + w_+[x(t,\mathrm{d}t),t,\mathrm{d}t]$ and $V_-[x(t,\mathrm{d}t),t,\mathrm{d}t] = v_-[x(t),t] + w_-[x(t,\mathrm{d}t),t,\mathrm{d}t]$, which can be decomposed in terms of differentiable parts v_+ and v_-, and of fractal parts w_+ and w_-. Note that, contrary to other attempts such as Nelson's stochastic quantum mechanics which introduces forward and backward velocities [9], the two velocities are here both forward, since they do not correspond to a reversal of the time coordinate, but of the time differential element now considered as an independent variable.

Going back to differentials, the elementary displacements $\mathrm{d}X$ on the geodesics of a nondifferentiable space can therefore be decomposed as the sum of two terms [2, 3, 5] (we omit the three-dimensional indices for simplicity)

$$\mathrm{d}_\pm X = \mathrm{d}_\pm x + \mathrm{d}_\pm \xi, \quad (5)$$

$\mathrm{d}_\pm \xi$ representing the 'fractal (differentiable) part' and $\mathrm{d}_\pm x$, the 'classical (non-differentiable) part', defined as

$$\mathrm{d}_\pm x = v_\pm \,\mathrm{d}t, \quad (6)$$

$$\mathrm{d}_\pm \xi = \eta_\pm \sqrt{2\mathcal{D}}\,\mathrm{d}t^{1/2}, \quad (7)$$

where η is a normalized stochastic (or simply fluctuating) variable such that $\langle \eta \rangle = 0$ and $\langle \eta^2 \rangle = 1$. This expression for the fractal fluctuation $\mathrm{d}_\pm \xi$ corresponds to the fractal dimension $D_F = 2$.

Then one combines the two time derivatives in terms of a complex derivative operator [2]

$$\frac{\widehat{\mathrm{d}}}{\mathrm{d}t} = \frac{1}{2}\left(\frac{\mathrm{d}_+}{\mathrm{d}t} + \frac{\mathrm{d}_-}{\mathrm{d}t}\right) - \frac{\mathrm{i}}{2}\left(\frac{\mathrm{d}_+}{\mathrm{d}t} - \frac{\mathrm{d}_-}{\mathrm{d}t}\right). \quad (8)$$

Applying this operator to the position vector yields a complex velocity

$$\mathcal{V} = \frac{\widehat{\mathrm{d}}}{\mathrm{d}t} x(t) = V - \mathrm{i}U = \frac{v_+ + v_-}{2} - \mathrm{i}\frac{v_+ - v_-}{2}. \quad (9)$$

One finds that the complex time derivative operator reads [2, 7, 10]

$$\frac{\widehat{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \widehat{\mathcal{V}} \cdot \nabla, \quad (10)$$

where $\widehat{\mathcal{V}} = \mathcal{V} - \mathrm{i}\mathcal{D}\nabla$. It plays the role of a 'covariant derivative operator', i.e., of a tool that allows the equation of dynamics in a fractal nondifferentiable space to be given the same form as in a classical differentiable one.

Namely, the classical part of the system can be characterized by a Lagrange function $\mathcal{L}(x, \mathcal{V}, t)$, from which an action $\tilde{\mathcal{S}}$ is defined as $\tilde{\mathcal{S}} = \int_{t_1}^{t_2} \mathcal{L}(x, \mathcal{V}, t)\mathrm{d}t$, where both the Lagrange function and the action are now complex because the velocity $\mathcal{V}$ is itself complex. Using this covariant derivative, one then writes the equation of dynamics in a potential ϕ under Newton's classical form (although this equation is no longer classical)

$$m\frac{\widehat{\mathrm{d}}}{\mathrm{d}t}\mathcal{V} = -\nabla\phi. \tag{11}$$

In the case when there is no external field the covariance is explicit, since equation (11) can be identified with a geodesics equation that takes the form of Galileo's equation of inertial motion,

$$\frac{\widehat{\mathrm{d}}}{\mathrm{d}t}\mathcal{V} = 0. \tag{12}$$

This vacuum form of the motion equation is also obtained when the field can be itself derived from a covariant derivative under a geometric interpretation, as in the case of gravitation in Einstein's theory of general relativity of motion, and now of gauge fields [24].

In both cases (with and without external field), the complex momentum $\mathcal{P}$ reads $\mathcal{P} = m\mathcal{V}$ [2], so that the complex velocity $\mathcal{V}$ is the gradient of the complex action, $\mathcal{V} = \nabla\tilde{\mathcal{S}}/m$.

2.2. Derivation of a generalized Schrödinger equation

Then one introduces a complex function ψ which is nothing but another expression for the complex action $\tilde{\mathcal{S}}$,

$$\psi = \mathrm{e}^{\mathrm{i}\tilde{\mathcal{S}}/S_0}, \tag{13}$$

where one can show that the constant S_0, which must be introduced for dimensional reasons, is linked to the parameter $\mathcal{D}$ by the relation $S_0 = 2m\mathcal{D}$ [7]. In the case of standard quantum mechanics, S_0 is nothing else but the Planck constant, $S_0 = \hbar$, so that $\mathcal{D} = \hbar/2m$. But, as we shall see, all the mathematical structure of the derivation holds for any value of the constant $\mathcal{D}$, which defines here the amplitude of the fractal fluctuations (equation (7)). The ψ function is therefore related to the complex velocity as follows:

$$\mathcal{V} = -2\mathrm{i}\mathcal{D}\nabla\ln\psi, \tag{14}$$

so that the fundamental equation of dynamics of equation (11) reads

$$2\mathrm{i}m\mathcal{D}\frac{\widehat{\mathrm{d}}}{\mathrm{d}t}(\nabla\ln\psi) = \nabla\phi. \tag{15}$$

Replacing $\widehat{\mathrm{d}}/\mathrm{d}t$ by its expression, given by equation (10), and replacing $\mathcal{V}$ by its expression in equation (14), one obtains after some calculations the equation

$$\nabla\phi = \mathrm{i}2m\mathcal{D}\nabla\left[\frac{\partial\psi/\partial t - \mathrm{i}\mathcal{D}\Delta\psi}{\psi}\right], \tag{16}$$

which can finally be integrated under the form of a Schrödinger equation [2]

$$\mathcal{D}^2\Delta\psi + \mathrm{i}\mathcal{D}\frac{\partial\psi}{\partial t} - \frac{\phi}{2m}\psi = 0. \tag{17}$$

We recover the standard quantum-mechanical equation in the special case $\mathcal{D} = \hbar/2m$, but it is generalized here to any value of $\mathcal{D}$.

It is important here to remark that the parameter $\mathcal{D}$ is a mere re-expression, up to constants, of a fundamental length scale of the theory, $\lambda_C = 2\mathcal{D}/c$. Since standard quantum mechanics is recovered for $\mathcal{D} = \hbar/2m$, this length scale is nothing but a generalization of the Compton length, $\lambda_C = \hbar/mc$. This involves two consequences which are relevant for the interpretation of the suggested experiments.

(i) The Compton length receives in this framework a new interpretation, namely, it gives the amplitude of the fractal fluctuations. Indeed, this amplitude reads under the form $(\mathrm{d}\xi/\lambda_C)^2 = \eta^2(c\,\mathrm{d}t/\lambda_C)$, so that the Compton length (and therefore the mass in standard quantum theory) can be defined in a geometric way as

$$\lambda_C = \frac{\langle \mathrm{d}\xi^2 \rangle}{c\,\mathrm{d}t}, \tag{18}$$

where $\langle\rangle$ denotes averaging over the stochastic or fluctuating variable η.

(ii) The de Broglie length also acquires, in such a framework, a simple geometric interpretation as a classical to fractal transition (in scale space). Indeed, since $\mathrm{d}x$ and $\mathrm{d}t$ are differential elements of the same order, one may write $\mathrm{d}\xi^2 = \eta^2 \times \lambda_C c\,\mathrm{d}t$ under the form $\mathrm{d}\xi^2 = \eta^2 \times \lambda\,\mathrm{d}x$. The new length scale introduced in this expression for dimensional reasons therefore reads $\lambda_{\mathrm{dB}} = c\lambda_C/(\mathrm{d}x/\mathrm{d}t)$, i.e.,

$$\lambda_{\mathrm{dB}} = \frac{c}{v} \times \lambda_C, \tag{19}$$

which generalizes the non-relativistic de Broglie scale since this yields $\lambda_{\mathrm{dB}} = \hbar/mv$ when $\lambda_C = \hbar/mc$ (standard quantum case). The elementary displacements therefore read $\mathrm{d}X = \mathrm{d}x + \mathrm{d}\xi = \mathrm{d}x(1 + \eta\sqrt{\lambda_{dB}/\mathrm{d}x})$, and they indeed show a transition from a classic, differentiable behavior to a fractal, nondifferentiable (i.e., scale-divergent) behavior when $\mathrm{d}x$ becomes smaller than λ_{dB}.

This last result allows one to identify $\mathcal{D}$ in a given physical situation, since in most cases the observable characteristic length scale of a system coming under such equations is the de Broglie scale (while λ_C is in general far too small, because of the factor v/c).

This description can be generalized to the many-particle case [18]. Indeed, the fractality parameter $\mathcal{D}$ does not characterize directly the fractal spacetime itself, but its geodesics. It is therefore quite possible to define a unique spacetime (in terms of a unique action $\tilde{\mathcal{S}}$ and therefore of a unique wavefunction $\psi = \mathrm{e}^{\mathrm{i}\tilde{\mathcal{S}}/\hbar}$) which contains different sub-sets of geodesics with different geometric properties corresponding to different particles. This is reminiscent of Einstein's general relativity where various bodies with different active gravitational masses M_i and therefore different Schwarzschild radii $2GM_i/c^2$, each of them following their own geodesics, contribute to the same unique spacetime.

For example, in the two-particle case, two different velocity fields of geodesics, $\mathcal{V}_1 = -2\mathrm{i}\mathcal{D}_1\nabla_1 \ln\psi$ and $\mathcal{V}_2 = -2\mathrm{i}\mathcal{D}_2\nabla_2 \ln\psi$ can be defined for a single wavefunction $\psi(x_1, y_1, z_1, x_2, y_2, z_2, t)$. The Lagrange function keeps the form of the classical Lagrange function for two particles, but now in terms of these complex velocity fields, i.e., $\mathcal{L} = (1/2)m_1\mathcal{V}_1^2 + (1/2)m_2\mathcal{V}_2^2 - \phi$. Then the complex Hamilton function, which was found to read $\mathcal{H} = \widehat{\mathcal{V}}\cdot\mathcal{P} - \mathcal{L}$ in the one-particle case [7], becomes for two particles $\mathcal{H} = \widehat{\mathcal{V}}_1\cdot\mathcal{P}_1 + \widehat{\mathcal{V}}_2\cdot\mathcal{P}_2 - \mathcal{L}$, and the Hamilton–Jacobi energy equation $\mathcal{H} + \partial\tilde{\mathcal{S}}/\partial t = 0$ becomes $\mathcal{H}\psi = \mathrm{i}S_0\partial\psi/\partial t$. The calculation is the same as for the one-particle case [7] except for the existence of two terms instead of one, so that, after replacing the complex velocity field by its expression in terms of ψ, one finally derives a two-particle Schrödinger equation,

$$S_0\left[(\mathcal{D}_1\Delta_1 + \mathcal{D}_2\Delta_2)\psi + \mathrm{i}\frac{\partial\psi}{\partial t}\right] = \phi\psi. \tag{20}$$

Another generalization of such an equation will be given in what follows (equation (124)). One recovers the two-particle Schrödinger equation of standard quantum mechanics for $\mathcal{D}_1 = \hbar/2m_1$, $\mathcal{D}_2 = \hbar/2m_2$ and $S_0 = \hbar$.

For a more complete development of the application of this approach to standard quantum mechanics (which has been only summarized here since it is not the direct subject of the present paper), see [7] in particular for a full derivation of the basic 'postulates' of quantum mechanics, and [11] for the question of the origin and universality of the Planck constant $\hbar$ in microphysics.

Now, the generality of this description also involves the possibility to build macroscopic quantum-type systems or devices which are no longer constrained by the Planck constant (see section 6)—for example, systems embedded in fractal media which would be scaling on a large range of scales [2], or artificial devices in which a quantum potential would be simulated by a classical one—and/or the possibility that such systems do already exist in nature [2, 4, 10, 12, 13, 19].

3. Schrödinger equation in fluids mechanics

3.1. From Schrödinger to Euler and continuity equations

By separating the real and imaginary parts of the generalized Schrödinger equation and by using a mixed representation of the motion equations in terms of (P, V), instead of (V, U) in the geodesics form and (P, θ) in the Schrödinger form, one obtains fluid dynamics-like equations, i.e., an Euler equation and a continuity equation (this is a generalization of the Madelung–Bohm transformation, but whose physical meaning is set from the very beginning instead of being *a posteriori* interpreted).

Let us recall explicitly this transformation. We first come back to the definition of the wavefunction by making explicit the probability and the phase, namely,

$$\psi = \sqrt{P} \times \mathrm{e}^{\mathrm{i}S/2m\mathcal{D}}. \tag{21}$$

By introducing this form of the wavefunction in the Schrödinger equation (17) with an exterior scalar potential ϕ, we obtain

$$\left\{-\frac{\sqrt{P}}{2m}\left(\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + \phi - 2m\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}}\right) + \mathrm{i}\frac{\mathcal{D}}{2\sqrt{P}}\left(\frac{\partial P}{\partial t} + \mathrm{div}\left(P\frac{\nabla S}{m}\right)\right)\right\}\mathrm{e}^{\mathrm{i}S/2m\mathcal{D}} = 0. \tag{22}$$

Now the complex velocity $\mathcal{V} = V - \mathrm{i}U$ is being linked to the wavefunction by the relation $\mathcal{V} = -2\mathrm{i}\mathcal{D}\nabla \ln \psi$, its real part is therefore given in terms of the phase by the standard classical relation [2]

$$V = \frac{\nabla S}{m}. \tag{23}$$

We note once again that this fundamental identification is here derived (since V has been defined from the very beginning as the real part of the geodesics mean velocity field), while in the standard Madelung transformation V is defined from the above equation itself, and it is therefore interpreted from it. In the scale relativity/nondifferentiable spacetime approach, the velocity field and therefore the wavefunction from which it derives characterize from the beginning the bundle of potential fractal geodesics.

By replacing, in the above form of the Schrödinger equation, $\nabla S/m$ by the real velocity field V, it reads

$$\left\{-\frac{\sqrt{P}}{2m}\left(\frac{\partial S}{\partial t}+\frac{1}{2}mV^2+\phi+Q\right)+\mathrm{i}\frac{\mathcal{D}}{2\sqrt{P}}\left(\frac{\partial P}{\partial t}+\mathrm{div}(PV)\right)\right\}\mathrm{e}^{\mathrm{i}S/2m\mathcal{D}}=0. \tag{24}$$

The real part of this equation is an energy equation,

$$E=-\frac{\partial S}{\partial t}=\frac{1}{2}mV^2+\phi+Q, \tag{25}$$

whose gradient yields an Euler-type equation

$$\left(\frac{\partial}{\partial t}+V\cdot\nabla\right)V=-\nabla\left(\frac{\phi+Q}{m}\right), \tag{26}$$

while the imaginary part is a continuity equation, namely,

$$\frac{\partial P}{\partial t}+\mathrm{div}(PV)=0. \tag{27}$$

But, in the energy and Euler equations, an additional potential energy Q has emerged, that writes

$$Q=-2m\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{28}$$

This scalar potential generalizes to a constant $\mathcal{D}$ which may be different from $\hbar/2m$ the quantum potential obtained in the Madelung–Bohm transformation. The potential Q is now understood as a manifestation of the fractal geometry, and the probability density is also interpreted in this framework as arising from the distribution of geodesics, so that the Born postulate is derived [5, 7, 17], as can be verified by numerical simulations [27, 28].

3.2. Inverse derivation: from Euler to Schrödinger equation (pressure-less potential motion)

It is less well known that the transformation from the generalized Schrödinger equation to the Euler and continuity equations with quantum potential is reversible. Indeed, the Euler and continuity system reads in the pressureless case

$$\left(\frac{\partial}{\partial t}+V\cdot\nabla\right)V=-\nabla\left(\phi-2\mathcal{D}^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \tag{29}$$

$$\frac{\partial\rho}{\partial t}+\mathrm{div}(\rho V)=0. \tag{30}$$

Their form is similar to equations (26) and (27), but with the probability density P replaced by the matter density ρ and with $m=1$. Assume, as a first step, that the motion is irrotational (see section 3.5 for the account of vorticity and pressure). Then we set

$$V=\nabla S. \tag{31}$$

Equation (29) takes the successive forms

$$\frac{\partial}{\partial t}(\nabla S)+\frac{1}{2}\nabla(\nabla S)^2+\nabla\left(\phi-2\mathcal{D}^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)=0, \tag{32}$$

$$\nabla\left(\frac{\partial S}{\partial t}+\frac{1}{2}(\nabla S)^2+\phi-2\mathcal{D}^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)=0, \tag{33}$$

which can be integrated as

$$\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 + \phi + C - 2\mathcal{D}^2 \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0, \tag{34}$$

where C is a constant that can be taken to be zero by a redefinition of the potential energy ϕ. Let us now combine this equation with the continuity equation as follows:

$$\left\{ -\frac{1}{2}\sqrt{\rho}\left(\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 + \phi - 2\mathcal{D}^2 \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) + \mathrm{i}\frac{\mathcal{D}}{2\sqrt{\rho}}\left(\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \nabla S)\right)\right\} \mathrm{e}^{\mathrm{i}S/2\mathcal{D}} = 0. \tag{35}$$

We have therefore recovered the form (22) of the Schrödinger equation (with $m = 1$). Finally we set

$$\psi = \sqrt{\rho} \times \mathrm{e}^{\mathrm{i}S/2\mathcal{D}}, \tag{36}$$

and equation (35) is strictly identical to the following generalized Schrödinger equation:

$$\mathcal{D}^2 \Delta\psi + \mathrm{i}\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\phi}{2}\psi = 0. \tag{37}$$

Given the linearity of the equation obtained, one can normalize the modulus of ψ by replacing the matter density ρ by a probability density $P = \rho/M$, where M is the total mass of the fluid in the volume considered. These two representations are equivalent.

The imaginary part of this generalized Schrödinger equation amounts to the continuity equation and its real part to the energy equation that reads

$$E = -\frac{\partial S}{\partial t} = \frac{1}{2} m V^2 + \phi - 2\mathcal{D}^2 \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \tag{38}$$

3.3. From Euler to Schrödinger: account of pressure

Consider now the Euler equations with a pressure term and a quantum potential term:

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right) V = -\nabla\phi - \frac{\nabla p}{\rho} + 2\mathcal{D}^2 \nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \tag{39}$$

When $\nabla p/\rho = \nabla w$ is itself a gradient, which is the case of an isentropic fluid, and, more generally, of every cases when there is a univocal link between pressure and density, e.g., a state equation [31], its combination with the continuity equation can be still integrated in terms of a Schrödinger-type equation [4],

$$\mathcal{D}^2 \Delta\psi + \mathrm{i}\mathcal{D}\frac{\partial}{\partial t}\psi - \frac{\phi + w}{2}\psi = 0. \tag{40}$$

Now the pressure term needs to be specified through a state equation, which can be chosen as taking the general form $p = k_p \rho^\gamma$.

In particular, in the sound approximation, the link between pressure and density writes $p - p_0 = c_s^2(\rho - \rho_0)$, where c_s is the sound speed in the fluid, so that $\nabla p/\rho = c_s^2 \nabla \ln \rho$. In this case, which corresponds to $\gamma = 1$, we obtain the nonlinear Schrödinger equation

$$\mathcal{D}^2 \Delta\psi + \mathrm{i}\mathcal{D}\frac{\partial}{\partial t}\psi - k_p \psi \ln|\psi| = \frac{1}{2}\phi\psi, \tag{41}$$

with $k_p = c_s^2$. When $\rho - \rho_0 \ll \rho_0$, one may use the additional approximation $c_s^2 \nabla \ln \rho \approx \left(c_s^2/\rho_0\right)\nabla\rho$, and the equation obtained takes the form of the nonlinear Schrödinger equation which is encountered in the study of superfluids and of Bose–Einstein condensates (see e.g. [32, 33]), and which is also similar to the Ginzburg–Landau equation of superconductivity [35] (here in the absence of field),

$$\mathcal{D}^2\Delta\psi + \mathrm{i}\mathcal{D}\frac{\partial}{\partial t}\psi - \beta|\psi|^2\psi = \frac{1}{2}\phi\psi, \tag{42}$$

with $\beta = c_s^2/2\rho_0$. In the highly compressible case the dominant pressure term is rather of the form $p \propto \rho^2$, so that $p/\rho \propto \rho = |\psi|^2$, and one still obtains a nonlinear Schrödinger equation of the same kind [32].

3.4. From the Schrödinger equation in a vectorial field to Euler and continuity equations

Let us now consider a more general case. In the previous sections, only a scalar external field was taken into account. We shall now study the decomposition of the Schrödinger equation which applies to a system subjected to a vectorial field (such as, e.g., a magnetic field). As we shall now show, it can also be generally decomposed in terms of an Euler-type equation and a continuity-type equation, with the external vectorial field playing a role similar to the rotational part of the velocity field. Thanks to this analogy, this decomposition applies actually to two different cases: (i) quantum fluids subjected to a magnetic field (such as in the Ginzburg–Landau equation of superconductivity) and (ii) some fluids with a non-potential velocity field.

Start from the general form of the Schrödinger equation for a spinless particle subjected to a scalar field ϕ and to a vectorial field K_j (for example, an electromagnetic field):

$$\left\{\frac{1}{2}(-2\mathrm{i}\mathcal{D}\nabla - K)^2 + \frac{\phi}{m}\right\}\psi = 2\mathrm{i}\mathcal{D}\frac{\partial\psi}{\partial t}. \tag{43}$$

In order to prepare the reverse derivation in which K actually represents the rotational part of the velocity field of the fluid under consideration, we have given here to the potential K a form in which it has the dimensionality of a velocity. In the case of an electromagnetic field, it is related to the vector potential A by the relation $K = (e/mc)A$. In the particular case when $\mathcal{D} = \hbar/2m$, one recovers the Schrödinger equation of standard quantum mechanics in the presence of a vectorial field,

$$\left\{\frac{1}{2m}(-\mathrm{i}\hbar\nabla - mK)^2 + \phi\right\}\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}. \tag{44}$$

Note that this equation may itself be found from the scale relativistic interpretation of gauge field theories according to which the field and the charges emerge as manifestations of the fractality of spacetime [24, 25, 30]. In this approach, the QED covariant derivative $-\mathrm{i}\hbar\tilde{\nabla} = -\mathrm{i}\hbar\nabla - mK$ can be derived from geometric first principles, and therefore the electromagnetic Schrödinger equation can be established as the integral of a geodesic equation (see also [26]).

Let us expand the Hamiltonian. We obtain (reintroducing for clarity indices running from 1 to 3)

$$-2\mathcal{D}^2\Delta\psi + 2\mathrm{i}\mathcal{D}K_k\partial^k\psi + \mathrm{i}\mathcal{D}(\partial_k K^k)\psi + \frac{1}{2}(K_kK^k)\psi + \frac{\phi}{m}\psi = 2\mathrm{i}\mathcal{D}\frac{\partial\psi}{\partial t}. \tag{45}$$

We now express the wavefunction ψ in terms of its modulus and of its phase,

$$\psi = \sqrt{P} \times \mathrm{e}^{\mathrm{i}\theta}. \tag{46}$$

Therefore we have

$$\partial_k\psi = (\partial_k\sqrt{P} + \mathrm{i}\sqrt{P}\partial_k\theta)\,\mathrm{e}^{\mathrm{i}\theta}, \qquad \partial_t\psi = (\partial_t\sqrt{P} + \mathrm{i}\sqrt{P}\partial_t\theta)\,\mathrm{e}^{\mathrm{i}\theta}, \tag{47}$$

$$\Delta\psi = \{(\partial_k\partial^k\sqrt{P} - \sqrt{P}\partial_k\theta\partial^k\theta) + \mathrm{i}(2\partial_k\theta\partial^k\sqrt{P} + \sqrt{P}\partial_k\partial^k\theta)\}\,\mathrm{e}^{\mathrm{i}\theta}. \tag{48}$$

The Schrödinger equation becomes, after simplification of the $\mathrm{e}^{\mathrm{i}\theta}$ term in the factor,

$$\begin{aligned}&-2\mathcal{D}^2(\partial_k\partial^k\sqrt{P} - \sqrt{P}\partial_k\theta\partial^k\theta) - 2\mathcal{D}\sqrt{P}K_k\partial^k\theta + \left(\frac{1}{2}K_kK^k + \frac{\phi}{m}\right)\sqrt{P} + 2\mathcal{D}\sqrt{P}\partial_t\theta\\ &\quad + \mathrm{i}\{-2\mathcal{D}^2(\sqrt{P}\partial_k\partial^k\theta + 2\partial_k\theta\partial^k\sqrt{P})\\ &\quad + 2\mathcal{D}K_k\partial^k\sqrt{P} + \mathcal{D}(\partial_kK^k)\sqrt{P} - 2\mathcal{D}\partial_t\sqrt{P}\} = 0.\end{aligned} \tag{49}$$

3.4.1. Continuity equation. Let us first consider the imaginary part of this equation. After multiplication by $2\sqrt{P}$ it becomes

$$-2\mathcal{D}\partial_tP - 2\mathcal{D}^2(2P\Delta\theta + 2\partial_kP\partial^k\theta) + 2\mathcal{D}(K_k\partial^kP + P\partial_kK^k) = 0. \tag{50}$$

Without the indices, it reads

$$\partial_tP + 2\mathcal{D}(P\Delta\theta + \nabla P\cdot\nabla\theta) - K\cdot\nabla P - P\nabla\cdot K = 0. \tag{51}$$

Let us now introduce, as in the scalar field case, a potential motion velocity field

$$V = 2\mathcal{D}\nabla\theta. \tag{52}$$

We obtain

$$\partial_tP + P\nabla\cdot V - P\nabla\cdot K + \nabla P\cdot V - \nabla P\cdot K = 0. \tag{53}$$

This leads us to define a full 'velocity field' as

$$v = V - K, \tag{54}$$

in terms of which the above equation reads

$$\partial_tP + P\nabla\cdot v + \nabla P\cdot v = 0, \tag{55}$$

and finally becomes the continuity equation

$$\frac{\partial P}{\partial t} + \mathrm{div}(Pv) = 0, \tag{56}$$

which is therefore generally valid, provided it is written in terms of the full velocity field $v = V - K$ instead of only the velocity field V (which is linked to the phase of the wavefunction).

3.4.2. Energy equation. Let us now consider the real part of equation (49). It reads

$$\sqrt{P}\left[\left(2\mathcal{D}\partial_t\theta + 2\mathcal{D}^2\partial_k\theta\partial^k\theta - 2\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}} + \frac{\phi}{m}\right) - 2\mathcal{D}K_k\partial^k\theta + \frac{1}{2}K_kK^k\right] = 0. \tag{57}$$

We now use the equivalent notation $S = 2m\mathcal{D}\theta$ ($= \hbar\theta$ in the case of standard quantum mechanics), so that the wavefunction is now defined, like in previous sections, as

$$\psi = \sqrt{P}\times\mathrm{e}^{\mathrm{i}S/2m\mathcal{D}}. \tag{58}$$

We obtain

$$\sqrt{P}\left[\partial_tS + \frac{1}{2m}\partial_kS\partial^kS + \phi - 2m\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}} - K_k\partial^kS + \frac{1}{2}mK_kK^k\right] = 0. \tag{59}$$

The potential part of the full velocity field now reads

$$V = \frac{\nabla S}{m}, \tag{60}$$

and we get the energy equation

$$\frac{\partial S}{\partial t} + \frac{1}{2} m V^2 + \frac{1}{2} m K^2 - m V \cdot K + \phi - 2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}} = 0. \tag{61}$$

One recognizes, once again, the emergence of the full velocity field $v = V - K$ in this equation, where V is potential while K is rotational. In its terms the energy equation takes the same form as in the scalar field case, namely,

$$-\frac{\partial S}{\partial t} = \frac{1}{2} m v^2 + \phi - 2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{62}$$

When the energy is conserved, $E = -\partial S/\partial t$. We therefore recover the same three contributions of the kinetic energy $E_c = \frac{1}{2} m v^2$, exterior potential energy ϕ, and quantum potential energy

$$Q = -2m\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}, \tag{63}$$

as in the previous case. The quantum potential also keeps exactly its previous form in this new (vectorial field) situation.

Let us now take the gradient of the energy equation. One obtains

$$\frac{\partial V}{\partial t} + \frac{1}{2} \nabla(v^2) = -\nabla\left(\frac{\phi + Q}{m}\right). \tag{64}$$

In the potential case, $\frac{1}{2}\nabla(v^2) = (v \cdot \nabla)v$. But here, in the case of the rotational motion, this relation leads to the introduction of a vorticity-like quantity, $\omega = \text{curl}\, v$, i.e.,

$$\omega_{\alpha k} = \partial_\alpha v_k - \partial_k v_\alpha = \partial_k K_\alpha - \partial_\alpha K_k. \tag{65}$$

Since K represents here a vector potential, $-\omega = \text{curl}\, K$ therefore represents a magnetic-like field. In tensorial notation we have

$$\tfrac{1}{2}\partial_\alpha(v^k v_k) = v^k \partial_\alpha v_k = v^k \partial_k v_\alpha + v^k(\partial_\alpha v_k - \partial_k v_\alpha) = v^k \partial_k v_\alpha + v^k \omega_{\alpha k}, \tag{66}$$

i.e.,

$$\tfrac{1}{2}\nabla(v^2) = (v \cdot \nabla)v + v \times \omega. \tag{67}$$

Therefore, since $V = v + K$ and $\text{curl}\, v = -\text{curl}\, K$, one finally obtains the equation

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\frac{\partial K}{\partial t} + v \times \text{curl}\, K - \nabla\left(\frac{\phi + Q}{m}\right). \tag{68}$$

One recognizes in the right-hand side of this equation the exact analog of a Lorentz force, to which is added the quantum force $-\nabla Q/m$. The term $-\partial K/\partial t$ is the analog of the magnetic contribution $-\partial A/c\partial t$ to the electric field $\mathcal{E} = -\partial A/c\partial t - \nabla\phi$, while $v \times \text{curl}\, K$ is the analog of the magnetic force $(e/c)v \times \text{curl}\, A$ (see e.g. [29]).

This equation has therefore exactly the form of the Euler equation that is expected for a fluid of the velocity field v coupled to a scalar potential ϕ and to a vectorial potential K, and subjected to an additional quantum potential Q. It agrees with the continuity equation which is also written in terms of the full velocity field v.

3.4.3. From the Ginzburg–Landau equation to fluid equations with magnetic field and quantum potential. Such an approach can be applied to the transformation of the Ginzburg–Landau equation of superconductivity into the classical equations for a fluid subjected to a magnetic field and to a quantum-like potential.

Let us start indeed from the Ginzburg–Landau equation of superconductivity [35] generalized to a coefficient $\mathcal{D}$ which may be different from $\hbar/2$,

$$\left(\mathcal{D}\nabla - \mathrm{i}\frac{K}{2}\right)^2 \psi + \alpha\psi - \beta|\psi|^2\psi = 0, \tag{69}$$

where $A = (mc/e)K$ is the magnetic vector potential.

From the previous decomposition, it is equivalent to the classical continuity and Euler equations of a fluid subjected both to a Lorentz force and to a quantum potential Q, namely (for $m = 1$)

$$\frac{\partial P}{\partial t} + \mathrm{div}(Pv) = 0, \tag{70}$$

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{\partial K}{\partial t} + v \times \mathrm{curl}\, K - \nabla Q, \tag{71}$$

where $P = |\psi|^2$ and

$$Q = -2\mathcal{D}^2 \frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{72}$$

The reversibility of the transformation (see section 3.5) means that, if one applies to a classical charged fluid a classical force having exactly the form of the 'quantum potential' Q (with a coefficient $\mathcal{D}$ no longer limited to the microscopic value $\hbar/2$), such a fluid would be described by the Ginzburg–Landau equation and it would therefore acquire some of the properties of a superconductor.

3.4.4. Euler equation when $v \times \mathrm{curl}\, v$ vanishes. A more simple form of Euler equation may be recovered in rather general situations, as we shall now see.

When $v \times \mathrm{curl}\, v = 0$, this means that v and $\mathrm{curl}\, v$ are parallel, i.e., $\mathrm{curl}\, v = \lambda v$ (Beltrami stream). In this case the Schrödinger in vectorial field equation takes the form of a standard Euler and continuity system of equations for a fluid subjected to a quantum-type potential $Q = -2m\mathcal{D}^2\Delta\sqrt{P}/\sqrt{P}$ and to a force $F_K = -\partial K/\partial t$, namely,

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = F_K - \nabla\left(\frac{\phi + Q}{m}\right), \tag{73}$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(Pv) = 0. \tag{74}$$

3.4.5. Euler equation when $v \times \mathrm{curl}\, v$ is a gradient. When $v \times \mathrm{curl}\, v = \nabla\xi_f/m$,which corresponds to $\mathrm{curl}(v \times \mathrm{curl}\, v) = 0$, ξ_f plays the role of an additional scalar potential, and the Schrödinger equation in the vectorial field may also be given the form of a standard Euler and continuity system of equations for a fluid subjected to a quantum-type force $F_Q = -\nabla Q$, with $Q = -2m\mathcal{D}^2\Delta\sqrt{P}/\sqrt{P}$, to a force $F_K = -\partial K/\partial t$, and to a total force $F = -\nabla(\xi_f + \phi)/m$, namely,

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = F_K - \nabla\left(\frac{\phi + \xi_f + Q}{m}\right), \tag{75}$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(P v) = 0. \tag{76}$$

3.5. Inverse problem: from Euler equation with vorticity to Schrödinger equation with a vectorial field

The previous calculations are reversible in the case when $v \times \mathrm{curl}\, v$ is a gradient, and therefore they allow us to achieve a new result. Namely, the equations of the motion of a fluid including a rotational component subjected to a quantum-type potential can also be integrated in terms of a (possibly nonlinear) Schrödinger equation, the rotational part of the motion appearing in it under the same form as an external vectorial field.

Consider a classical non-viscous fluid subjected to a scalar potential ϕ and described by its velocity field $v(x, y, z, t)$ and its density $\varrho(x, y, z, t)$. These physical quantities are solutions of the Euler and continuity equations,

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v = -\nabla\phi - \frac{\nabla p}{\varrho}, \tag{77}$$

$$\frac{\partial \varrho}{\partial t} + \mathrm{div}(\varrho v) = 0. \tag{78}$$

In the case of an isoentropic fluid, and more generally in all cases when there exists a univocal link between the pressure p and the density ϱ, $\nabla p/\varrho$ becomes a gradient [31], namely $\nabla p/\varrho = \nabla w$, where w is the enthalpy by mass unit in the isentropic case ($s = \mathrm{cst}$). In this case we set

$$\nabla\phi + \frac{\nabla p}{\varrho} = \nabla(\phi + w) = \nabla\Phi, \tag{79}$$

and the Euler equation becomes

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v = -\nabla\Phi. \tag{80}$$

Let us now assume that the classical fluid is subjected to an additional force

$$F_Q = -\nabla Q = 2\mathcal{D}^2 \nabla \left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right), \tag{81}$$

so that the Euler and continuity equations read

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v = -\nabla\left(w + \phi + Q\right), \tag{82}$$

$$\frac{\partial \varrho}{\partial t} + \mathrm{div}(\varrho v) = 0. \tag{83}$$

The Euler equation can be written in the form

$$\frac{\partial v}{\partial t} + \frac{1}{2}\nabla(v^2) - v \times \mathrm{curl}\, v = -\nabla(w + \phi + Q). \tag{84}$$

In this section, we specifically consider the case when the velocity field v is no longer potential. However, we can decompose it in terms of a potential (irrotational) contribution V and a rotational one K. Namely, we set

$$v = V - K, \qquad V = \nabla S. \tag{85}$$

Then we build a 'wavefunction' from the potential part only, by combining this function S and the density ϱ in terms of a complex function:

$$\psi = \sqrt{\varrho} \times e^{iS/2\mathcal{D}}. \tag{86}$$

Therefore $\partial v/\partial t = \partial \nabla S/\partial t - \partial K/\partial t = \nabla(\partial S/\partial t) - \partial K/\partial t$, so that the Euler equation now reads

$$\frac{\partial K}{\partial t} - v \times \text{curl}\, K = \nabla\left(\frac{\partial S}{\partial t} + \frac{1}{2}v^2 + w + \phi + Q\right). \tag{87}$$

The scalar expression under the gradient is not vanishing in the general case. We call $-\chi(x, y, z, t)$ this function, and we may therefore write a (formal) generalized energy equation,

$$-\frac{\partial S}{\partial t} = \frac{1}{2}v^2 + w + \phi + Q + \chi, \tag{88}$$

while equation (87) now writes

$$\frac{\partial K}{\partial t} - v \times \text{curl}\, K = -\nabla\chi. \tag{89}$$

We have now recovered the conditions (energy equation and continuity equation) which lead to the construction of a nonlinear (NL) Schrödinger-type equation in terms of a complex linear combination of these two equations.

Therefore the whole calculation of section 3.4 can be reversed in this case (with $m = 1$ and $P \propto \varrho$), so that we can integrate the Euler and continuity system in terms of a nonlinear Schrödinger-type equation including a vectorial field (analogous to the standard Schrödinger equation of a charged particle in a magnetic field),

$$\left(\mathcal{D}\nabla - i\frac{K}{2}\right)^2 \psi + i\mathcal{D}\frac{\partial\psi}{\partial t} = \left(\frac{w + \phi + \chi}{2}\right)\psi, \tag{90}$$

where we recall that $\psi = \sqrt{\varrho} \times \exp(iS/2\mathcal{D})$. In general the pressure, and therefore the enthalpy w is a function of the density $\rho = |\psi|^2$, which contributes to the nonlinearity of this equation. In the absence of vorticity, it is similar to the kind of NL Schrödinger equation encountered in the study of superfluids and Bose–Einstein condensates (see e.g. [32, 33]).

Equation (90) can also be given an expanded form,

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial\psi}{\partial t} = \left\{\frac{\phi + w + \chi}{2} + \frac{K^2}{4} + i\frac{\mathcal{D}}{2}\nabla\cdot K + i\mathcal{D}K\cdot\nabla\right\}\psi, \tag{91}$$

where the term between brackets in the right-hand side may be interpreted, when $K \cdot \nabla\psi$ is negligible, as a generalized potential energy.

In this Schrödinger equation, the rotational part K of the velocity field $v = V - K$ plays the role of an external vector potential, and therefore the vorticity $\omega = -\text{curl}\, K$ the role of the corresponding field. Its evolution equation is obtained by taking the curl of the Euler equation, namely,

$$\frac{\partial\omega}{\partial t} = \text{curl}(v \times \omega), \tag{92}$$

which is equivalent to equation (89), but without the unknown function χ.

However, the situation here remains different and more complicated than the quantum-mechanical Schrödinger equation in an external electro-magnetic field of electric potential ϕ and vectorial potential K (up to constants), which is accompanied by the Maxwell equations for the external field. Here two terms are added, (i) the pressure p, expressed in terms of the enthalpy w, which may be known as a function of the density ρ, i.e. $|\psi|^2$, through a state equation and lead to a nonlinear contribution and (ii) the unknown function $\chi(x, y, z, t)$. Therefore, except for the large class of flows for which $\chi = \text{cst}$, i.e.,

$$\frac{\partial K}{\partial t} - v \times \text{curl}\, K = 0, \tag{93}$$

in the general case this system of equations remains incomplete, since an equation for χ is lacking. Nevertheless, despite this situation, this result may remain physically meaningful and useful, since the Schrödinger equation and its solutions have general properties which are valid whatever the applied fields. It shows that the application of a quantum-like potential on a fluid is sufficient to transform the energy and continuity equations into a Schrödinger-like equation for a function ψ linked to the density by $\rho = |\psi|^2$, even in the presence of vorticity.

Let us finally consider the stationary version of equation (90) in the general case when the pressure terms read $w = p/\rho \propto \rho$ (see section 3.3). We obtain

$$\left(\mathcal{D}\nabla - \mathrm{i}\frac{K}{2}\right)^2 \psi + \alpha\psi - \beta|\psi|^2\psi = 0, \tag{94}$$

with $\alpha = (E - \phi - \chi)/2$. This equation has exactly the form of the Ginzburg–Landau equation of superconductivity [35], generalized to a coefficient $2\mathcal{D}$ which may be different from $\hbar$. This result may be applied to the two cases initially considered at the beginning of section 3.4, namely:

(i) The case where K represents the true vector potential of a magnetic field. It may correspond to a classical charged fluid subjected to an electromagnetic field and to a classical potential which has been tuned in order to give it the form of the quantum potential Q. As already remarked in section 3.4.3, the equations of motion of such a fluid (continuity equation and Euler equation with a Lorentz force) may be combined in terms of a single complex equation which takes the form of the Ginzburg–Landau equation of superconductivity. One may therefore hope such a fluid to acquire some of the properties of a quantum fluid.

(ii) The case when K does not represent here an external magnetic field, but a rotational part of the velocity field. This means that a nonlinear Schrödinger form can also be given to the equation of motion of fluids showing vorticity and subjected to an external potential Q having a quantum-like form.

3.6. From Navier–Stokes to the nonlinear Schrödinger equation

Let us finally consider the general case of Navier–Stokes equations including a viscosity term. The fluid mechanics equations including a quantum-type potential read in this case

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v = \nu \Delta v - \frac{\nabla p}{\varrho} - \nabla(\phi + Q), \tag{95}$$

$$\frac{\partial \varrho}{\partial t} + \text{div}(\varrho v) = 0, \tag{96}$$

where the quantum-type potential energy is still given by

$$Q = -2\mathcal{D}^2 \frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}. \tag{97}$$

We set as in the previous section

$$v = V - K, \qquad V = \nabla S, \qquad \psi = \sqrt{\varrho} \times \mathrm{e}^{\mathrm{i}S/2\mathcal{D}}, \tag{98}$$

i.e., V is the potential part of the full velocity field v. Therefore the viscosity term reads

$$\nu\Delta v = \nu\Delta(\nabla S - K) = \nu\nabla(\Delta S) - \nu\Delta K, \tag{99}$$

and, assuming once again $\nabla p/\varrho = \nabla w$, the Navier–Stokes equation now takes the form

$$-\frac{\partial K}{\partial t} + \nu\Delta K - v \times \mathrm{curl}\, v = -\nabla\left(\frac{\partial S}{\partial t} - \nu\Delta S + \frac{1}{2}v^2 + w + \phi + Q\right). \tag{100}$$

This equation is not generally integrable. However, it nevertheless becomes integrable for a large class of flows, namely, those for which its left-hand side is a gradient,

$$-\frac{\partial K}{\partial t} + \nu\Delta K - v \times \mathrm{curl}\, v = \nabla\chi. \tag{101}$$

In this case, one obtains an energy and a continuity equation that read

$$\frac{\partial S}{\partial t} - \nu\Delta S + \frac{1}{2}v^2 + w + \phi + \chi + Q = 0, \qquad \frac{\partial\varrho}{\partial t} + \mathrm{div}(\varrho v) = 0, \tag{102}$$

and which can be combined into the form of a nonlinear Schrödinger equation of the magnetic type,

$$\left(\mathcal{D}\nabla - \mathrm{i}\frac{K}{2}\right)^2\psi + \mathrm{i}\mathcal{D}\frac{\partial\psi}{\partial t} = \frac{1}{2}\left(w + \phi + \chi - \nu\Delta S\right)\psi. \tag{103}$$

The viscosity therefore leads to add a new nonlinear term in this NL Schrödinger equation that depends on the phase $S/2\mathcal{D}$ of the wavefunction. When the fluid motion is irrotational, the integration under the form of a NL Schrödinger equation of the continuity and Navier–Stokes equations including a quantum potential is always possible.

4. A Schrödinger equation for the rotational motion of a solid

4.1. Introduction

In the previous sections, a Schrödinger form has been obtained for the equations of motion and of continuity of a fluid subjected to a quantum potential. However, the method we used may be applied not only to a fluid but also to a mechanical system. Indeed, we have shown [4] that the scale relativity approach can be applied to the rotational motion of a solid, leading once again to a Schrödinger-type equation. Here we give an improved demonstration of this Schrödinger equation, then, as in previous sections, decompose it in terms of its real and imaginary parts, and then obtain a new generalized form of the quantum potential. Conversely, the addition of such a new quantum potential in the energy equation yields, in combination with the continuity equation, a Schrödinger equation.

4.2. Equation of rotational solid motion in scale relativity

Let us briefly recall the results of [4, 36], in which a Schrödinger form was obtained for the equation of the rotational motion of a solid subjected to the three basic effects of a fractal and nondifferentiable space (namely, infinity of trajectories, fractal dimension 2 and reflection symmetry breaking of the time differential element).

The role of the variables (x, v, t) of translational motion is now played respectively by (φ, Ω, t), where φ stands for three rotational position angles (for example, Euler angles) and Ω for the angular velocity. We choose a contravariant notation φ^k for the angles, where the indices run from 1 to 3, and we adopt Einstein's convention for summation on upper and lower indices. The Euler–Lagrange equations for rotational motion classically write [34]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \Omega^k} = \frac{\partial L}{\partial \varphi^k}, \tag{104}$$

in terms of a Lagrange function $L = (1/2)I_{ik}\Omega^i\Omega^k - \Phi$, where I_{ik} is the tensor of inertia of the solid body and Φ its potential energy in an exterior field. Therefore the angular momentum of the system is

$$M_i = \frac{\partial L}{\partial \Omega^i} = I_{ik}\Omega^k. \tag{105}$$

The torque is given by

$$K_i = \frac{\partial L}{\partial \varphi^i} = -\frac{\partial \Phi}{\partial \varphi^i} \tag{106}$$

and the motion equations finally take the Newtonian form

$$\frac{\mathrm{d}M_i}{\mathrm{d}t} = K_i. \tag{107}$$

Let us now consider the generalized description of such a system in the scale relativity framework. Following the same road as for position coordinates, in the generalized situation when spacetime is fractal, the angle differentials $\mathrm{d}\varphi = \mathrm{d}x_\varphi + \mathrm{d}\xi_\varphi$ can be decomposed in terms of two contributions, a classical (differentiable) part $\mathrm{d}x_\varphi$ and a fractal fluctuation $\mathrm{d}\xi_\varphi$ which is such that $\langle \mathrm{d}\xi_\varphi \rangle = 0$ and

$$\left\langle \mathrm{d}\xi_\varphi^j\, \mathrm{d}\xi_\varphi^k \right\rangle = 2\mathcal{D}^{jk}\,\mathrm{d}t, \tag{108}$$

where $\mathcal{D}^{jk}$ is now a tensor which generalizes the scalar parameter $\mathcal{D}$ of the translational case. As we shall see in the following, this tensor is, up to a multiplicative constant, similar to a metric tensor.

The breaking of reflexion invariance ($\mathrm{d}t \leftrightarrow -\mathrm{d}t$) on the time differential elements, which is a consequence of the nondifferentiability, yields a two-valuedness of the angular velocity [2, 3, 5]. This leads to introducing a complex angular velocity $\widetilde{\Omega}$, then a complex Lagrange function $\widetilde{L}(\varphi, \widetilde{\Omega}, t)$. The two effects of nondifferentiability and fractality of space can finally be combined in terms of a rotational quantum-covariant derivative [4],

$$\frac{\widehat{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \widetilde{\Omega}^k\partial_k - \mathrm{i}\mathcal{D}^{\mathrm{jk}}\partial_{\mathrm{j}}\partial_{\mathrm{k}}, \tag{109}$$

where $\partial_k = \partial/\partial\varphi^k$. Using this quantum-covariant derivative, we may generalize to fractal motion the equation of rotational motion while keeping its classical form,

$$I_{jk}\frac{\widehat{\mathrm{d}}\,\widetilde{\Omega}^k}{\mathrm{d}t} = -\partial_j\Phi, \tag{110}$$

where I_{jk} is the tensor of inertia of the solid and Φ an externally added potential.

We then introduce a complex function (which will subsequently be identified with a wavefunction) as another expression for the complex action $\widetilde{S} = \int \widetilde{L}\,\mathrm{d}t$,

$$\psi = \mathrm{e}^{\mathrm{i}\widetilde{S}/S_0}, \tag{111}$$

where S_0 is a real constant introduced for dimensional reasons. Now the complex angular momentum is, like in classical solid mechanics, linked to the complex action by the standard relation $\widetilde{M}_k = \partial\widetilde{S}/\partial\varphi^k$, so that one obtains

$$\widetilde{M}_k = I_{\alpha k}\widetilde{\Omega}^k = -\mathrm{i}S_0\partial_\alpha \ln\psi. \tag{112}$$

Let us therefore introduce the inverse of the tensor of inertia, $[I]^{-1} = I^{\alpha k}$, such that

$$I_{\alpha k} I^{k\beta} = \delta_\alpha^\beta. \tag{113}$$

This allows us to express the complex velocity field in terms of the wavefunction,

$$\widetilde{\Omega}^k = -\mathrm{i}S_0 I^{k\alpha} \partial_\alpha \ln \psi. \tag{114}$$

We can now replace the velocity field by this expression in the covariant derivative and in the rotational motion equations. We obtain

$$-\mathrm{i}S_0 \left(\frac{\partial}{\partial t} - \mathrm{i}S_0 I^{k\beta} \partial_\beta \ln \psi \, \partial_k - \mathrm{i}\mathcal{D}^{jk} \partial_j \partial_k \right) \partial_\alpha \ln \psi = K_\alpha, \tag{115}$$

which can be written as

$$-\mathrm{i}S_0 \left(\partial_\alpha \frac{\partial}{\partial t} \ln \psi - \mathrm{i}\{S_0 \, \partial_\beta \ln \psi \; I^{k\beta} \partial_k \partial_\alpha \ln \psi + \mathcal{D}^{jk} \partial_j \partial_k \partial_\alpha \ln \psi\} \right) = K_\alpha. \tag{116}$$

We have reversed in the second equation the places of $I^{k\beta}$ and ∂_β: this is possible since $I^{k\beta}$ is assumed to be constant. This reversal allows one to make the operator $I^{k\beta}\partial_k$ appear. Provided the tensor of inertia plays the role of a metric tensor, we have $I^{k\beta}\partial_k = \partial^\beta$, and we recognize in the expression under brackets {} a tensorial generalization of the expression which was encountered in the translational motion case, namely,

$$S_0 m^{-1} (\partial_\beta \ln \psi \, \partial^\beta) \partial_\alpha \ln \psi + \mathcal{D} \partial_k \partial^k \partial_\alpha \ln \psi. \tag{117}$$

Indeed, as recalled at the beginning of this paper, the relation

$$S_0 m^{-1} = 2\mathcal{D}, \tag{118}$$

that is nothing but a generalized Compton relation, allows one to transform this expression into a remarkable identity which leads to the integration of the motion equation in terms of a Schrödinger equation.

Now we are able to generalize this result to the rotational motion case, despite the complication brought by the fact that the mass is replaced by the inertia tensor. Indeed, the inverse of the mass is replaced by the inverse tensor $[I]^{-1}$, and we can identify the fractal fluctuation tensor with this metric tensor up to a constant, namely

$$\mathcal{D}^{\alpha\beta} = \frac{S_0}{2} I^{\alpha\beta}, \tag{119}$$

i.e., in matrix form, $S_0[I]^{-1} = 2[\mathcal{D}]$. (Note the correction to [4, 36] where an inverse relation between these quantities was erroneously given; the Schrödinger equation obtained in these papers nevertheless remains correct). This is a new tensorial generalization of the Compton relation. Moreover this means that it is the inertia tensor itself which serves as a metric tensor and can be used to raise and lower the indices, e.g., $I^{jk}\partial_j\partial_k = \partial^k\partial_k$, while $\mathcal{D}^{jk}$ does the same but up to a constant, namely, $\mathcal{D}^{jk}\partial_j\partial_k = (S_0/2)\partial^k\partial_k$.

The existence of a similarity between the rotational diffusion term $\widehat{M}_j D^{jk} \widehat{M}_k$, where $\widehat{M}$ denotes angular momentum operators, and the corresponding quantum-mechanical Hamiltonian $\widehat{M}_j I^{jk} \widehat{M}_k/2$ of a rigid body has already been remarked by Dale Favro [37] in his theory of rotational Brownian motion. Here we directly identify $\mathcal{D}^{jk}$ to $(S_0/2)I^{jk}$ (but $\mathcal{D}^{jk}$, despite its stochastic definition, should not be confused with a standard diffusion coefficient).

The equation of motion now takes the form

$$-\mathrm{i}S_0 \left(\partial_\alpha \frac{\partial}{\partial t} \ln \psi - \mathrm{i}\frac{S_0}{2} \{I^{k\beta} \partial_\beta \ln \psi \, \partial_k \partial_\alpha \ln \psi + I^{jk} \partial_j \partial_k \partial_\alpha \ln \psi\} \right) = K_\alpha, \tag{120}$$

and, using the tensorial notation $I^{jk}\partial_j = \partial^k$, it can now be written as

$$-\mathrm{i}S_0\left(\partial_\alpha \frac{\partial}{\partial t}\ln\psi - \mathrm{i}\frac{S_0}{2}\{2\partial^k \ln\psi\, \partial_k\partial_\alpha \ln\psi + \partial^k\partial_k\partial_\alpha \ln\psi\}\right) = K_\alpha. \tag{121}$$

This expression can be simplified under the form

$$-\mathrm{i}S_0\left(\partial_\alpha \frac{\partial}{\partial t}\ln\psi - \mathrm{i}\frac{S_0}{2}\partial_\alpha \frac{\partial_k\partial^k\psi}{\psi}\right) = -\partial_\alpha\Phi, \tag{122}$$

and finally be written globally as a gradient,

$$\partial_\alpha S_0\left\{\frac{(S_0/2)\partial_k\partial^k\psi + \mathrm{i}\partial\psi/\partial t}{\psi}\right\} = \partial_\alpha\Phi. \tag{123}$$

This equation can therefore be integrated in the general case under the form of a new generalized Schrödinger equation that reads [4, 36]

$$S_0\left(\mathcal{D}^{jk}\partial_j\partial_k\psi + \mathrm{i}\frac{\partial}{\partial t}\psi\right) = \Phi\psi. \tag{124}$$

In terms of the inverse tensor of inertia this rotational Schrödinger equation reads

$$\frac{1}{2}S_0^2 I^{jk}\partial_j\partial_k\psi + \mathrm{i}S_0\frac{\partial}{\partial t}\psi = \Phi\psi. \tag{125}$$

Since the tensor of inertia plays the role of a metric tensor, in particular for the lowering and raising of indices, it can finally be written as

$$\frac{1}{2}S_0^2\, \partial^k\partial_k\psi + \mathrm{i}S_0\frac{\partial}{\partial t}\psi = \Phi\psi, \tag{126}$$

which keeps the form of the scalar case [2], while generalizing it.

The standard quantum case is recovered by identifying S_0 with $\hbar$, but, once again, all the mathematical structure of the equation (and therefore of its solutions) is preserved with a constant that can have any value, including a macroscopic one.

We may now conclude by returning to the fractal angular fluctuations that writes in terms of the inverse inertia tensor

$$\langle \mathrm{d}\xi_\varphi^j\, \mathrm{d}\xi_\varphi^k\rangle = S_0 I^{jk}\,\mathrm{d}t. \tag{127}$$

We therefore gain a complete justification of the identification of the tensor of inertia with a metric tensor, since, owing to the fact that $I^{kj}I_{jk} = \delta_k^k = 3$, we obtain the invariant metric relation

$$\mathrm{d}t = \frac{I_{jk}}{3S_0}\langle \mathrm{d}\xi_\varphi^j\, \mathrm{d}\xi_\varphi^k\rangle, \tag{128}$$

where $S_0 = \hbar$ in the standard quantum case, and where $\mathrm{d}t$ (which appears instead of its square $\mathrm{d}t^2$ as an expression of the fractal dimension 2) is indeed the fundamental invariant here since all this study is done in the framework of Galilean motion relativity.

4.3. Fluid representation and newly generalized quantum potential

Let us now give this Schrödinger equation a fluid mechanical form. This can be easily done by following the same steps as in section 3.1, but now using tensorial derivative operators. We set

$$\psi = \sqrt{P} \times \mathrm{e}^{\mathrm{i}S/S_0}, \tag{129}$$

and we replace ψ by this expression in equation (126). The imaginary part of this equation reads

$$\frac{1}{2}\sqrt{P}\partial_k\partial^k S + \partial_k S\partial^k\sqrt{P} + \frac{\partial\sqrt{P}}{\partial t} = 0. \tag{130}$$

i.e., after simplification,

$$\frac{\partial P}{\partial t} + \partial_k(P\partial^k S) = 0. \tag{131}$$

Since S is the real part of the complex action, it is linked to the angular momentum M^k (which is itself the real part of the complex angular momentum) and to the real angular velocity Ω^j by the relations

$$M_\alpha = I_{\alpha j}\Omega^j = \partial_\alpha S. \tag{132}$$

Now, since $I^{k\alpha}I_{\alpha j} = \delta^k_j$, we find

$$\Omega^k = I^{k\alpha}I_{\alpha j}\Omega^j = I^{k\alpha}M_\alpha. \tag{133}$$

Therefore

$$\Omega^k = I^{k\alpha}\partial_\alpha S = \partial^k S. \tag{134}$$

Finally, we find that the imaginary part of the rotational Schrödinger equation amounts, once again, to a continuity equation in the general case

$$\frac{\partial P}{\partial t} + \partial_k(P\Omega^k) = 0. \tag{135}$$

Note the correction to [4, 36], in which we concluded that this was the case only in some particular reference systems. This means that the probability interpretation of $P = |\psi|^2$ is also generally ensured [5, 17].

The real part of equation (126) takes the form

$$\frac{\partial S}{\partial t} + \Phi - \frac{1}{2}S_0^2\frac{\partial_k\partial^k\sqrt{P}}{\sqrt{P}} + \frac{1}{2}\partial_k S\partial^k S = 0. \tag{136}$$

Let us first consider the last term of this expression. It reads

$$\tfrac{1}{2}\partial_k S\partial^k S = \tfrac{1}{2}M_k\Omega^k = \tfrac{1}{2}I_{jk}\Omega^j\Omega^k = T_{\rm rot}, \tag{137}$$

which is the classical expression of the rotational kinetic energy. In the conservative case, $E = -\partial S/\partial t$ is the total energy, so that we recover the standard energy equation,

$$E = \Phi + Q + T_{\rm rot}, \tag{138}$$

but which now includes an additional potential energy that reads

$$Q = -S_0\frac{\mathcal{D}^{jk}\partial_j\partial_k\sqrt{P}}{\sqrt{P}} = -\frac{1}{2}S_0^2\frac{I^{jk}\partial_j\partial_k\sqrt{P}}{\sqrt{P}}. \tag{139}$$

This is a new generalization of the quantum potential in the rotational case.

An Euler-like equation including this quantum potential is simply obtained by taking the gradient of this equation, namely,

$$\partial_\alpha\left(\frac{\partial S}{\partial t}\right) + \partial_\alpha\left(\frac{1}{2}\Omega^k M_k\right) = -\partial_\alpha(\Phi + Q). \tag{140}$$

Now one has $\partial_\alpha(\partial S/\partial t) = \partial(\partial_\alpha S)/\partial t = \partial M_\alpha/\partial t$, and since $M_\alpha = \partial_\alpha S$ is a gradient, $\partial_\alpha T_{\rm rot} = \Omega^k\partial_\alpha M_k = \Omega^k\partial_k M_\alpha$, so that we finally obtain

$$\left(\frac{\partial}{\partial t} + \Omega^k\partial_k\right)M_\alpha = -\partial_\alpha(\Phi + Q), \tag{141}$$

which is indeed the expected generalization in terms of an Euler equation of the equation of dynamics $\mathrm{d}M/\mathrm{d}t = K$ in the case when the rotational velocity becomes a velocity field, namely, $\Omega = \Omega[\varphi(t), t]$.

4.4. From an Euler to a Schrödinger equation

Reversely, one may now consider a rotating body or an ensemble of rotating bodies which are subjected to a fluctuating motion of rotation, such that the rotational velocity can be replaced, at least as an approximation, by a rotational velocity field $\Omega = \Omega[\varphi(t), t]$.

Assume, moreover, that each body is subjected, in addition to the torque $-\partial_\alpha \Phi$ of an exterior field, to a quantum-like torque $K_Q = -\nabla_\varphi Q$, where the quantum potential Q is given by equation (139).

Such a system would be described by an Euler equation and a continuity equation,

$$\left(\frac{\partial}{\partial t} + \Omega^k \partial_k\right) M_\alpha = -\partial_\alpha \left(\Phi - \frac{1}{2} S_0^2 \frac{I^{jk} \partial_j \partial_k \sqrt{P}}{\sqrt{P}}\right), \tag{142}$$

$$\frac{\partial P}{\partial t} + \partial_k (P \Omega^k) = 0, \tag{143}$$

which, after introducing the wavefunction $\psi = \sqrt{P} \times \mathrm{e}^{\mathrm{i}S/S_0}$, can be recombined to yield a generalized Schrödinger equation that reads

$$\frac{1}{2} S_0^2 I^{jk} \partial_j \partial_k \psi + \mathrm{i} S_0 \frac{\partial}{\partial t} \psi = \Phi \psi, \tag{144}$$

so that it would be expected to show some kind of quantum-type properties. Indeed, in the particular case $S_0 = \hbar$, we recover the standard Schrödinger equation of the quantum-mechanical description of a rigid body which is used, e.g., for determining the rotational levels of molecules taken as a whole. In the macroscopic case, such an equation has been applied with positive results to the study of the probability distribution of the inclination and obliquity of chaotic astronomical bodies [13, 36].

5. Diffusion potential opposite to the quantum potential

The question of the relation of the quantum theory with diffusion processes has been posed for long. This domain of research includes proposals according to which the quantum behavior may originate in a diffusion process, such as stochastic mechanics [9] (even though further works have shown that it corresponds to no existing classical diffusion [41, 42]) or binary random walks [43]. The relation between scale covariance, the Schrödinger equation and the hydrodynamical picture of diffusion-type processes has also been recently studied in [44, 45].

Our aim in this section is not to study in detail this question, but to enlighten it by a new result, according to which a diffusion process may also be written in terms of an Euler equation including an additional potential energy, which is exactly the *opposite* of a quantum potential. Such a result leads to characterize the quantum behavior, which is in many cases self-organizing and stabilizing (as demonstrated by the existence of stationary solutions describing stable structures such as atoms and molecules), as an opposite of the diffusion behavior, which is instead linked to entropy increase and, most of the time, disorganization.

Let us consider a classical diffusion process. Such a process is described by the Fokker–Planck equation:

$$\frac{\partial P}{\partial t} + \mathrm{div}(P v) = D \Delta P, \tag{145}$$

where D is the diffusion coefficient. When there is no global motion of the diffusing fluid or particles ($v = 0$), the Fokker–Planck equation is reduced to the usual diffusion equation for the probability P:

$$\frac{\partial P}{\partial t} = D\Delta P. \tag{146}$$

This well-known equation holds for the propagation of heat (in this case P is replaced by the temperature), for the diffusion of a fluid in a mixing of fluids (in this case P is replaced by the concentration of the diffusing fluid), and for the Brownian motion of particles diffusing in a fluid.

Conversely, when the diffusion coefficient vanishes, the Fokker–Planck equation is reduced to the continuity equation,

$$\frac{\partial P}{\partial t} + \mathrm{div}(Pv) = 0. \tag{147}$$

5.1. *Continuity equation*

Let us now make the change of variable:

$$V = v - D\nabla \ln P. \tag{148}$$

Let us first prove that, in the general case $v \neq 0$, the new velocity field $V(x, y, z, t)$ is now the solution of the standard continuity equation. Indeed, we obtain, by replacing V by its expression,

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = \frac{\partial P}{\partial t} + \mathrm{div}(Pv) - D\,\mathrm{div}(P\nabla \ln P) = \frac{\partial P}{\partial t} + \mathrm{div}(Pv) - D\Delta P. \tag{149}$$

Finally using the Fokker–Planck equation, we find

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0. \tag{150}$$

5.2. *Euler equation for $v = 0$*

Let us now establish the form of the Euler equation for the new velocity field V. Let us calculate its total time derivative, at first in the simplified case $v = 0$:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} + V\cdot\nabla\right)V = -D\frac{\partial}{\partial t}\nabla \ln P + D^2(\nabla \ln P\cdot\nabla)\nabla \ln P. \tag{151}$$

Now, since $\partial\nabla \ln P/\partial t = \nabla\partial \ln P/\partial t = \nabla(P^{-1}\partial P/\partial t)$, we can make use of the diffusion equation so that we obtain

$$\left(\frac{\partial}{\partial t} + V\cdot\nabla\right)V = -D^2\left(\nabla\frac{\Delta P}{P} - (\nabla \ln P\cdot\nabla)\nabla \ln P\right). \tag{152}$$

We shall now use the remarkable identity [7],

$$\frac{1}{\alpha}\nabla\left(\frac{\Delta R^\alpha}{R^\alpha}\right) = \Delta(\nabla \ln R) + 2\alpha(\nabla \ln R\cdot\nabla)(\nabla \ln R). \tag{153}$$

By using it for $R = P$ and $\alpha = 1$, we can replace $\nabla(\Delta P/P)$ by $\Delta(\nabla \ln P) + 2(\nabla \ln P\cdot\nabla)\nabla \ln P$, so that equation (152) becomes

$$\left(\frac{\partial}{\partial t} + V\cdot\nabla\right)V = -D^2\,[\Delta(\nabla \ln P) + (\nabla \ln P\cdot\nabla)\nabla \ln P]\,. \tag{154}$$

The right-hand side of this equation comes again under the identity (153), but now for $\alpha = 1/2$. Therefore we finally obtain the following form for the Euler equation of the velocity field V:

$$\left(\frac{\partial}{\partial t} + V \cdot \nabla\right) V = -2D^2 \nabla \left(\frac{\Delta\sqrt{P}}{\sqrt{P}}\right). \tag{155}$$

This result calls for the following comments:

(i) It gives an equivalence between a standard fluid subjected to a force field and a diffusion process.

(ii) The above 'diffusion force' derives from a potential $\phi_{\text{diff}} = 2D^2\Delta\sqrt{P}/\sqrt{P}$. This expression introduces a square root of probability in the description of a classical diffusion process.

(iii) This 'diffusion potential' is exactly the opposite of the quantum potential $Q/m = -2\mathcal{D}^2\Delta\sqrt{P}/\sqrt{P}$.

The relation between quantum-type processes and diffusion processes is now enlightened in a new way: they appear as exactly opposite, so that quantum-type behavior can be considered as an 'anti-diffusion' process in this context.

The change of the sign of the potential has therefore dramatic consequences, since in one case it yields a classical diffusion equation which is known to lead to disorganization, irreversibility and spreading in $\sqrt{t}$ while in the other it yields a Schrödinger equation that allows stationary solutions and leads to structuring and self-organization.

5.3. Euler equation for $v \neq 0$

Let us now consider the general situation of a non-vanishing global velocity field v. In order to do this calculation we now introduce the indices in an explicit way. Equation (151) takes the form

$$\frac{\partial V^k}{\partial t} + V^j\partial_j V^k = \frac{\partial v^k}{\partial t} - D\partial^k\left(\frac{\partial P/\partial t}{P}\right) + (v^j - D\partial^j \ln P)\partial_j(v^k - D\partial^k \ln P). \tag{156}$$

Accounting for the Fokker–Planck equation it becomes

$$\begin{aligned}\frac{\partial V^k}{\partial t} + V^j\partial_j V^k = {}& \left(\frac{\partial v^k}{\partial t} + v^j\partial_j v^k\right) - D\partial^k\left(\frac{D\Delta P - \partial_j P v^j - P\partial_j v^j}{P}\right) \\ & - Dv^j\partial_j\partial^k \ln P - D\partial^j \ln P\,\partial_j v^k + D^2\partial^j \ln P\,\partial_j\partial^k \ln P.\end{aligned} \tag{157}$$

After some calculation one finally obtains

$$\frac{\partial V^k}{\partial t} + V^j\partial_j V^k = -2D^2\partial^k\left(\frac{\partial_j\partial^j\sqrt{P}}{\sqrt{P}}\right) + \frac{\mathrm{d}v^k}{\mathrm{d}t} + D\{\partial^k\partial_j v^j + (\partial^k v^j - \partial^j v^k)\partial_j \ln P\}. \tag{158}$$

In the case when v is potential, the last rotational term vanishes and the force in the right-hand side of this equation derives itself from a potential

$$\Phi = 2D^2\frac{\Delta\sqrt{P}}{\sqrt{P}} - D\Delta\varphi + \frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2, \tag{159}$$

where we have set $v = \nabla\varphi$. This is in particular the case of the scale-relativistic description (see section 2.1) where $v = v_+$ is potential. The quantum potential (plus possibly an external potential ϕ) can therefore be obtained in this case provided

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 - D\Delta\varphi = \phi - 4D^2\frac{\Delta\sqrt{P}}{\sqrt{P}}. \tag{160}$$

Under this condition, the Euler and continuity equations can be integrated under the form of a Schrödinger equation,

$$D^2 \Delta\psi + \mathrm{i}D\frac{\partial\psi}{\partial t} = \frac{1}{2}\phi\psi. \tag{161}$$

Therefore the possible values of the velocity field $v = v_+$ differ fundamentally between the two situations (quantum versus diffusion). In particular $v_+ = U + V = 0$ is excluded in the quantum case, since it leads to the standard diffusion equation. This explains how the Fokker–Planck equation can be common to the two processes, despite their fundamental antinomy.

6. Discussion: possible applications

The aim of this paper was mainly to set the theoretical basis for observational and experimental applications of generalized quantum-like potentials. Specific works will be devoted to each of these applications [38, 39] that we shall therefore only briefly discuss here.

(1) The concept of quantum potential has been applied for long to standard quantum mechanical systems coming under the Planck constant $\hbar$ [16], in which case its amplitude is exclusively given by $-\hbar^2/2m$. It reveals to be particularly useful for the study of quantum fluids [32], since it can be computed in this case in terms of the density of matter and not only of a density of probability.

(2) The new proposal here consists of looking for systems subjected to quantum-like potentials of the form $Q = -2m\mathcal{D}^2\Delta\sqrt{P}/\sqrt{P}$ and their various generalizations such as, e.g., the tensorial form equation (139), whose amplitude could now be macroscopic. The parameter $\mathcal{D}$ can now be specific of the given system and no longer be constrained by the relation $\mathcal{D} = \hbar/2m$: not only the dependence on $\hbar$ is relaxed, but also the inverse dependence on the inertial mass. Such systems, although they remain classical in many of their properties (they would in particular not come under indistinguishability of identical particles, entanglement and EPR paradox, Pauli exclusion principle, etc) would nevertheless be described, as we have shown here, by a generalized 'wavefunction' solution of a Schrödinger equation and satisfying the Born postulate $P = |\psi|^2$. Note also that this description would be valid strictly only in a subdomain of the system: due to classical effects such as viscosity, one does not expect a complete vanishing of ψ (and then of the density or of the fluid height), thus preventing the singularity one may encounter for the quantum potential on nodal lines or surfaces.

We can suggest the following (possibly non-exhaustive) applications of this case:

(2.1) Systems which would manifest, at least in an approximative way, conditions leading to the transformation and integration of the equations of motion into a Schrödinger-type equation.

Let us consider two approaches in which such a transformation is achieved and then analyze the conditions that underlie this transformation. The first, which involves a diffusion process, is Nelson's stochastic mechanics [9]. In this theory a probability density is introduced from the beginning, which is a solution of the standard Fokker–Planck equation (i.e., the forward Kolmogorov equation of standard diffusion theory), but also of a 'backward Fokker–Planck' equation in which the average backward velocity is different from the forward one. The problem, in the present context, is that this backward Fokker–Planck equation is incompatible with the backward Kolmogorov equation of standard diffusion [40] and that, as pointed out by many authors [4, 7, 41, 42], it corresponds to no existing classical diffusion process. Moreover, the form of the mean acceleration in stochastics mechanics must be postulated. As a consequence, a full justification seems to be lacking for the application of this non-standard diffusion process to macroscopic systems, even though such an application has been proposed, in particular as a description of the diffusion process in protoplanetary disks [46, 47].

A second approach is the scale-relativity theory of quantum spacetime. When it is applied to standard quantum mechanics in the microphysical realm [2, 7], one assumes a full nondifferentiability of the spacetime continuum which gives rise to the various new contributions to the covariant derivative (equation (10)) in terms of which the motion equation is written as a geodesic equation (equation (12)). However, when it is applied to macroscopic systems [2, section 7.2], [4, 12], the interpretation is different. Indeed, the same contributions to the covariant derivative (section 2) may be obtained, at least as an approximation, provided three conditions be fulfilled: (i) the number of possible trajectories is infinite or very large (which leads to a fluid-like description in terms of a velocity field); (ii) each of these trajectories is of fractal dimension 2 over a large enough range of scales (at least 10^4 to 10^5) which ensures that the relation $\langle d\xi^2\rangle = 2\mathcal{D}\,dt$ remains a valid approximation of the true fluctuation on this range; (iii) there is a fundamental irreversibility on timescales δt small with respect to the characteristic timescales of the system, allowing a breaking of the reflexion symmetry $\delta t \leftrightarrow -\delta t$ and a two-valuedness of the velocity field. To these three conditions one should add a Newtonian dynamics, since a Langevin regime, e.g., of the Brownian motion type (force proportional to velocity) does not allow to derive a Schrödinger equation. (For a discussion of the relation between stochastic mechanics and the fractal spacetime approach, see [2, 4, 12, 7].)

An analysis of these conditions leads to suggest (at least) two situations where they could be fulfilled:

(2.1.1) *Natural systems*: we have suggested [2, chapter 7.2], [3, 4] that some chaotic systems, studied at timescales larger than the temporal horizon of predictibility implied by the exponential divergences, may satisfy the above conditions and could therefore be described by a Schrödinger-type equation at these timescales. This suggestion has been applied to the formation of planetary systems [2, 19, 20, 12]—in this case one finds that $\mathcal{D} = GM/2c\alpha_g$, where M is the star mass and α_g is a 'gravitational coupling constant' [48, 19]—and more generally to the formation of structures in cosmology [3, 4, 13]. The use of a Schrödinger representation in cosmology has also been proposed more recently in [15], at least as a method of resolution of hydrodynamics equations (by making $\hbar \to 0$, as also proposed in [4]), and in [14], including the account of a quantum potential (also previously introduced in [12]). At mesoscopic scales, we have suggested that some aspects of living systems, which are also characterized by fractality, stochastic fluctuations and small timescale irreversibility, could also come under a similar description (in a yet different context) [21, 22].

(2.1.2) *Experimental/artificial systems or devices*: in a fractal medium achieved over a large enough range of scales (larger than $\approx 10^4$, the equation of propagation of particles in this medium (provided their dynamics be Newtonian and the above three conditions be fulfilled) could take a Schrödinger-like form [2].

(2.2) Systems subjected to a quantum-like potential Q or to a quantum-like force $F_Q = -\nabla Q$. These cases also include possible natural systems and man-made devices:

(2.2.1) *Natural systems*: a classical potential could, in a transient way, take at random the form of a quantum-like potential. This could happen, e.g., for systems subjected to a highly fluctuating external field, which would yield rare events coming under a Schrödinger regime. A possible example is the highly fluctuating combination of currents, wind and ground shape achieved in the ocean, which could lead to the appearance of freak waves [39]. In biology, such a random and transient emergence of a Schrödinger regime due to fluctuating environmental conditions could have been selected during evolution on the basis of the advantages it would bring to the system (self-organization, morphogenesis, non-dissipation, etc [10, 22]). In this application to living systems, the nodal surfaces corresponding to divergences of the quantum potential, instead of being a problem, may play an important and relevant role in the description of the 'biological fluid', which is indeed characterized by domains separated by walls (cells, organs, etc).

(2.2.2) *Artificial/experimental devices subjected to a classical force simulating a quantum-like force*: we have suggested [22, 49] to achieve such a new kind of quantum-like system in a laboratory experiment by applying a classical potential taking the form of a quantum potential to a classical fluid though a retro-active loop involving real time measurements of its density (compressible fluid) or of the height of its surface in a basin (incompressible fluid [39]). Numerical simulations of such an experiment have given encouraging results [38].

7. Conclusion and future prospect

In this paper, we have recalled that the continuity and Euler equations including a quantum potential can always be integrated and combined under the form of a linear (without pressure) or nonlinear (with pressure) Schrödinger equation (NLSE) when the fluid motion is irrotational.

In the case of the fluid motion including vorticity and therefore possibly turbulence, a Schrödinger form can also be obtained, at least in a formal way. Such an equation is of the 'magnetic' Schrödinger form, in which the vorticity field plays a role similar to that of an exterior electromagnetic field. We intend to generalize this result by considering the possibility to use more complete tools, such as spinorial, bispinorial or multiplet wavefunctions (see [50] for a recent attempt of implementation of this proposal), and a generalized description of the vectorial vorticity field, using, e.g., non-Abelian gauge field theory [24].

The same transformation also holds for a classical charged fluid subjected to an electromagnetic field to which one also applies a potential having the form of a quantum potential. Such a fluid is then described by a Ginzburg–Landau-like equation, and it is therefore expected to have at least some of the properties of a quantum fluid. This particularly interesting case will be specifically studied in more detail in future works, since it could lead to a new kind of macroscopic superconducting-like behavior.

The same method has been applied to the chaotic rotational motion of a solid, and a generalized tensorial form of the quantum potential has been obtained in this case. Finally, we have shown in this paper that, after a change of variable, the diffusion equation can also be given the form of a continuity and Euler system including an additional potential energy. Since this potential is exactly the opposite of a quantum potential, the quantum behavior may be considered, in this context, to be equivalent to a kind of anti-diffusion. Consequences for the inverse problem (of possible partial reversal of the motion of a diffusive fluid, see e.g. [51]) will be considered in future works.

Let us conclude by recalling that some numerical simulations of possible future experimental devices implementing this theoretical description have given encouraging results [38]. Such kind of devices, in which the applied potential depends on the knowledge of some internal measurable properties of the system (such as a density of matter, the height of the surface of a fluid [39] or a probability density) involves a retro-action loop which may be typical of living-like systems [22].

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